E-ISSN 2520-0429
ISSN 2519-1284

European Journal of Economics, Law and Social Sciences

Vol. 7 No. 3
October, 2023

Research Article

Numerical approximation in the application of Risk Parity with Conditional Value at Risk in case of mixed portfolios

Denis Veliu
Head of Banking Finance Department, Metropolitan University, Tirana, Albania

https://doi.org/10.2478/ejels-2023-0013

Abstract

The 2008 financial crisis has required new methods for portfolios diversification. In the same year, Maillard, Roncalli and Teiletche (2008) suggested a method that maximizes the diversification which is called Risk Parity or Equally weighted Risk Contribution strategy. The most common method to use the Risk Parity approach is to use the standard deviation as risk measure. In this paper, we describe a method to apply Risk Parity to the Expected shortfall or also known as Conditional Value at Risk using a numerical approximation from discrete historical observation. The expected shortfall can use the advantage of being a coherent measure, not to forget that it is also a convex measure, which is very useful in the optimization. Another advantage is that the Risk Parity approach doesn't need the estimation of the expected return as an input. Usually, the models that require the expected returns, such as the Markowitz, model have higher concentration in a smaller number of assets. This will bring a very high turnover and drawdown of the performance. The performance analysis in this paper is applied in mixed portfolios composed by stock, bonds and commodities. They show also how better this model performs in case of the crisis. We also identify not only the strong points but also the week points of these models.

Keywords: Numerical approximation, application, Risk Parity, Conditional Value, mixed portfolios.

1. Introduction

The high volatility of the market of after the COVID pandemic and the Russian Ukrainian Crisis has pushed the investors in finding other ways to diversify their financial portfolio, even more if they invest in commodities like grain or crude oil. Using models like the most famous Mean-Variance (Markowitz,1952) or the Expected shortfall will bring high concentration on a small number of assets and perform poorly in the out of sample period. These models are also based on the expected returns that are typically estimated based on historical data or financial models. However, these estimates are subject to uncertainty and can be influenced by various factors such as market conditions, economic events, and changes in investor sentiment. If the expected returns used in the portfolio model turn out to be inaccurate, it can lead to suboptimal portfolio allocations and performance. Portfolio models that heavily rely on expected returns can be highly sensitive to changes in those assumptions (Merton 1980). Small variations in expected returns can have a significant impact on portfolio allocations and outcomes. This sensitivity can make the portfolio model more susceptible to errors and less robust in different market conditions.

Black & Litterman (1990) uses a Bayesian approach to achieve and change the estimated returns. Today there are many advanced models for forecasting using different simulation techniques. For portfolio selection under Risk Parity method, the first idea was introduced by Qian (2005) and from it the idea of equally risk contribution in proportion of stock and bond to achieve the long-term risk premium. He shows that this method is more efficient than the portfolio created from 60 per cent stocks and 40 per cent bonds. The introduction as risk parity models was formulated by Maillard, Roncalli and Teiletche (2008) using the standard deviation as measure of risk. In their paper they proved the existence of the uniqueness of the portfolio, also that the riskiness of Risk Parity models is between the minimum variance and the uniform (or naïve portfolio).

Risk Parity approaches are frequently used to allocate the risk of a portfolio by decomposing the total portfolio risk into the risk contribution of each component in the same quantity. One of the main advantages of the Risk Parity approach is that it does not require the estimation of the expected returns.

In this paper we describe a numerical method to use the Risk Parity to the expected shortfall or Conditional Value at Risk $CVaR_\alpha(x)$. The $CVaR_\alpha(x)$ is a coherent and convex method, so it is possible to use the Euler decomposition for first order homogeneous functions (Artzner, 1999). This decomposition needs the calculation of the derivatives of risk measure. Stefanovits (2009) applies the equally risk contribution to the expected shortfall in case of standardized multivariate distribution, using a Gaussian kernel estimation. He implemented Risk Parity approach to Expected Shortfall assuming normally or t-student data in a parametric approach. In this paper we show that we can apply the Risk Parity model with $CVaR_\alpha(x)$ as a risk measure to any (real) return distribution. This is possible with approximation methods in the calculation of the partial derivatives of the Conditional Value at Risk (Tasche, 2000). To have a complete pattern, we compare the Risk Parity strategies with different risk measures (standard deviation and Conditional Value at Risk). The results are very similar but the time of computation of Risk Parity with Conditional Value at Risk is significantly shorter.

The algorithms for the calculation of the weights of the optimal portfolios are developed in Matlab 2022a©, running in a Windows 10 operating system, in a computer with processor Intel (R) Core (TM) i7-7500U CPU @ 2.70- 2.90 GHz and 12 GB of RAM that compute the algorithm in a very short time. The algorithm uses an interior point solution using limited number of iterations in the case of the Risk Parity model with $CVaR_{\alpha}(x)$.

2. Research method

In the process of the calculation of the weights with the Risk Parity using Conditional Value at Risk we need to calculate the existence of the partial derivatives of $CVaR_{\alpha}(x)$. For the consistence of the model, we need to impose some assumptions on the distribution of the random vector R = (r_1, r_2,.,,r_n) . When we create a portfolio with n assets, each weight x_i, in vector x = (x_1, x_2,..,,x_n), we need to present sufficient conditions for quantile of the portfolio return $X = R'x = \sum_{i=1}^{n} x_i r_i$ to be differentiable respect to the weights x_i. These conditions rely on the existence of a conditional probability density function (pdf) of the i-th asset return r_i given the others which is measured as follow:

In the process of the calculation of the weights with the Risk Parity using Conditional Value at Risk we need to calculate the existence of the partial derivatives of . For the consistence of the model, we need to impose some assumptions on the distribution of the random vector R = (r_1, r_2,.,,r_n) . When we create a portfolio with n assets, each weight x_i, in vector x = (x_1, x_2,..,,x_n), we need to present sufficient conditions for quantile of the portfolio return to be differentiable respect to the weights . These conditions rely on the existence of a conditional probability density function (pdf) of the i-th asset return given the others which is measured as follow:

$$r_{i,t+1} = \frac{P_{i,t+1} - P_{i,t}}{P_{i,t}}$$

Starting from the paper of Tasche (2000), the first problem to deal with is differentiating the quantile function $q_{\alpha}(X)$, and form that, the expression of $CVaR_{\alpha}(x)$ partial derivatives.

Definition 1 For the random vector R = (r_1, r_2,.....,r_n) , r_1 has a conditional density given (r_2,.....,r_n) if it exits a measurable function $\theta\colon \mathbb{R}^n \longrightarrow [0, \infty)$ such for that all $A \in \mathcal{B}(\mathbb{R})$ we have

$$P[r_1, \in A|\, r_2, \ldots\ldots, r_n] = \int_A \ \theta\,(u, r_2, \ldots\ldots, r_n) du$$

The existence of a joint pdf of R implies the existence of the conditional probability density function but not necessarily the other way is true.

Lemma 1 assumes that r_1 has a conditional density θ given (r_2,.....,r_n), where (r_1,r_2,.....,r_n) is a an $\mathbb{R}^n$-valued random vector. For any weight vector x = (x_1, x_2,.....,x_n) ∈ $\mathbb{R}\backslash\{0\} \times \mathbb{R}^{n-1}$, we have:

1. The random variable $X = \sum_{i=1}^{n} x_i r_i$ has a pdf given by the following absolutely

continuous functions

$f_X(u) = \frac{1}{x_1} E\left[\theta\left(\frac{u-\sum_{j=2}^{n} x_j r_j}{x_1}, r_2, \ldots\ldots, r_n\right)\right]$ (1)

2. if $f_X(u) > 0$ we have almost surely for i=2,.....,n, and for $u \in \mathbb{R}$

$E\left[r_i \mid \sum_{j=1}^{n} x_j r_j = \mathrm{u}\right] = \frac{E\left[r_1 \theta\left(\frac{1}{x_1}\left(u-\sum_{j=2}^{n} x_j r_j\right), r_2, \ldots\ldots, r_n\right)\right]}{E\left[\theta\left(\frac{1}{x_1}\left(u-\sum_{j=2}^{n} x_j r_j\right), r_2, \ldots\ldots, r_n\right)\right]}$ (1. a)

3. if $f_X(u) > 0$ we have almost surely for i=2,.....,n, and for $u \in \mathbb{R}$

$E\left[r_1 \mid \sum_{j=1}^{n} x_j r_j = \mathrm{u}\right] = \frac{E\left[\frac{u-\sum_{i=2}^{n} x_i r_i}{x_1} \theta\left(\frac{1}{x_1}\left(u-\sum_{j=2}^{n} x_j r_j\right), r_2, \ldots\ldots, r_n\right)\right]}{E\left[\theta\left(\frac{1}{x_1}\left(u-\sum_{j=2}^{n} x_j r_j\right), r_2, \ldots\ldots, r_n\right)\right]}$ (1. b)

The point 1 of the Lemma says that if there is a conditional density of r_1 given the other component, then subject of the condition $x \neq 0$ the distribution $X = R'x = \sum_{i=1}^{n} x_i r_i$
is absolutely continuous with a density of point 1.

Proof.

1. Consider $x_1 > 0$, then we can write:

$P[X \leq u] = E\left[1_{\{X \leq u\}}\right] = E\left[E\left[1_{\{X \leq u\}}\right] | r_2, \ldots\ldots, r_n\right] = E\left[\int_{-\infty}^{\frac{u-\sum_{i=2}^{n} x_i r_i}{x_1}} \theta(v, r_2, \ldots\ldots, r_n) dv\right] =$
$E\left[\int_{-\infty}^{u} \frac{1}{x_1} \theta\left(\frac{v-\sum_{i=2}^{n} x_i r_i}{x_1}, r_2, \ldots\ldots, r_n\right) dv\right] = \int_{-\infty}^{u} E\left[\frac{1}{x_1} \theta\left(\frac{v-\sum_{i=2}^{n} x_i r_i}{x_1}, r_2, \ldots\ldots, r_n\right)\right] dv =$
$\frac{1}{x_1} E\left[\theta\left(\frac{v-\sum_{i=2}^{n} x_i r_i}{x_1}, r_2, \ldots\ldots, r_n\right)\right]$ (Using Fubini Theorem in order to change the order of integration)

For $x_1 < 0$ we proceed in the same way.

2. $E\left[r_i \mid \sum_{j=1}^{n} x_j r_j = \mathrm{u}\right] = \frac{E\left[r_i 1_{\{X \leq u\}}\right]}{P[X=u]} = \lim_{\delta \to 0} \frac{\delta^{-1} E\left[r_i 1_{\{u<X<u+\delta\}}\right]}{\delta^{-1} P(u<X<u+\delta)} = \frac{\frac{\vartheta}{\vartheta u} E\left[r_i 1_{\{X \leq u\}}\right]}{f_X(u)}$,
where $f_X(u)$>0 (2)

Furthermore, we have:

$\frac{\partial}{\partial u} E\left[r_i 1_{\{X \leq u\}}\right] = \frac{\partial}{\partial u} E\left[E r_i\left[1_{\{X \leq u\}}\right] |, r_2, \ldots\ldots, r_n\right] = \frac{1}{x_1} E\left[r_i \theta\left(\frac{u-\sum_{i=2}^{n} x_i r_i}{x_1}, r_2, \ldots\ldots, r_n\right)\right]$ (3)

Substituting (1) and (3) in (2) we obtain (1.a)

3. We can write the expression (1.a) and obtain (1.b)

$E\left[r_i \mid \sum_{j=1}^{n} x_j r_j = u\right] = E\left[\frac{u-\sum_{i=2}^{n} x_i r_i}{x_1} \mid \sum_{j=1}^{n} x_j r_j = u\right]$

These are possible only for these assumptions of the conditional density θ.
For more see the work of Tasche (2000).

Assumptions:

1. For fixed $r_2, \ldots\ldots, r_n$, the mapping $t \mapsto (t, r_2, \ldots\ldots, r_n)$ is continuous in t.
2. The map $(t, x) \mapsto E\left[\theta\left(\frac{u-\sum_{i=2}^{n} x_i r_i}{x_1}, r_2, \ldots\ldots, r_n\right)\right]$ is finite value and continuous.
3. For i=2,...,n the mapping $E\left[r_i \theta\left(\frac{u-\sum_{i=2}^{n} x_i r_i}{x_1}, r_2, \ldots\ldots, r_n\right)\right]$ is finite value and continuous.

Applying these assumptions, Tasche (2000) gives the conditions for the partial differentiation with respect to the weights.

Theorem 1 We assume that the distribution of the returns is formulated to guarantee the conditional density of r_i given $r_2, \ldots\ldots, r_n$, satisfying the above Assumptions in some open set $H \subset \mathbb{R}\backslash\{0\} \times \mathbb{R}^{n-1}$ and that $f_X(q_\alpha(X)) > 0$. Then $q_\alpha(X)$ is partially differentiable with respect to each weight x_i as follows:

$$\frac{\partial q_\alpha(X)}{\partial x_i} = E[r_i | R'x = q_\alpha(X)]$$

Proof. Applying Lemma 1 the random variable $X = R'x = \sum_{i=1}^n x_i r_i$ has a continuous pdf conditional density of r_1 given $(r_2, \ldots\ldots, r_n)$ as follow

$$f_X(u) = \frac{1}{x_1} E\left[\theta\left(\frac{u - \sum_{j=2}^n x_j r_j}{x_1}, r_2, \ldots\ldots, r_n\right)\right] \forall x \; with \; x_i \geq 0$$

$\alpha = P[X \leq q_\alpha(X)] = E\left[\int_{-\infty}^{\frac{q_\alpha(X) - \sum_{i=2}^n x_i r_i}{x_1}} \theta(v, r_2, \ldots\ldots, r_n) dv\right]$ (4)

Differentiating expression (1.4) with respect x_i for *i*=2,..,*n,* we have:

$0 = \frac{1}{x_1} E\left[\theta\left(\frac{u - \sum_{j=2}^n x_j r_j}{x_1}, r_2, \ldots\ldots, r_n\right)\right] = f_X(u)$ (5)

Solving (5) for $\frac{\partial q_\alpha(X)}{\partial x_i}$ and applying the Lemma 1, we find the result of Theorem 1:

$$\frac{\partial q_\alpha(X)}{\partial x_i} = E[r_i | R'x = q_\alpha(X)]$$

Note that $VaR_\alpha(x) = -q_\alpha(X)$ then we can write:

$$\frac{\partial VaR_\alpha(x)}{\partial x_i} = -E[r_i | R'x = -VaR_\alpha(x)]$$

Applying to $VaR_\alpha(x)$ the Euler decomposition we have

$$VaR_\alpha(x) = \sum_{i=1}^n \frac{\partial VaR_\alpha(x)}{\partial x_i} = -\sum_{i=1}^n E[r_i | R'x = -VaR_\alpha(x)]$$

The calculation of the partial derivatives for the Value at Risk is needed for the partial derivatives of the Conditional Value at Risk. Indeed, be the definition of $CVaR_\alpha(x)$ (Uryasev, 2000) we have

$CVaR_\alpha(x) = \frac{1}{\alpha} \int_0^\alpha VaR_v(x) \mathrm{d}v$ (6)

Thus, using the Assumption 1 and differentiating (6) we obtain that:

$$\frac{\partial CVaR_\alpha(x)}{\partial x_i} = \frac{1}{\alpha} \int_0^\alpha \frac{\partial CVaR_\alpha(x)}{\partial x_i} \mathrm{d}v = -\frac{1}{\alpha} \int_0^\alpha E[r_i | -R'x = VaR_\alpha(x)] \mathrm{d}v =$$

$-\frac{1}{\alpha} \int_0^\alpha E[r_i | X = q_\alpha(X)] \mathrm{d}v = - E[r_i | X \leq -VaR_\alpha(x)]$ (7)

The same result starting from the Expected shortfall $ES_\alpha(x)$,which is equivalent to the $CVaR_\alpha(x)$, as Tasche (2000) and Stefanovits (2010) showed in their work, under the condition that $E[X^-] < \infty$.

The going to the definition of Risk Parity, the Total Risk contribution for each asset i of a portfolio is given by the following expression:

$$TRC_i^{CVaR_\alpha(x)}(x) = x_i \frac{\partial CVaR_\alpha(x)}{\partial x_i}$$

The expression in case of continuous returns distribution is the following:

$$TRC_i^{CVaR_\alpha(x)}(x) = -x_i E[r_i | X \le -VaR_\alpha(x)]$$

$$CVaR_\alpha(x) = \sum_{i=1}^{n} TRC_i^{CVaR_\alpha(x)}(x) = -\sum_{i=1}^{n} x_i E[r_i | X \le -VaR_\alpha(x)]$$

Since Risk Parity equalizes the total risk contributions:

$$TRC_i(x) = TRC_j(x) \; \forall \; i\,, j$$

The Risk Parity model can be formulated as the following optimization problem:

$$x^* = \arg\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(TRC_i(x) - TRC_j(x) \right)^2$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x \ge 0$$

We will apply Risk Parity to the standard deviation and to Conditional Value at Risk in equal conditions: same starting points in the algorithms and with no short selling or no possibility to leverage, using historical scenarios of assets returns.

For the approximation using times series of observation with weekly or daily data, where the *i*-th asset return r_i consist of *T* number outcomes rji with *i*=1,.....,n and *j*=1,...,T. For each portfolio $x \in \mathbb{R}^n$ where n is the number of assets in the market, the vector of the observed portfolio returns is $R_P = (r_{p1}, \dots\dots, r_{pT})$ where:

$r_{pj} = x' r^j$ with j=1,....,T where $r^j = (r_{j1}, \dots\dots, r_{jT})$.

For a high number of observation *T* with weekly or daily data, we can apply the Law of Large Numbers for the numerical approximation of the empirical distribution of the historical portfolio return:

$$P(R_P \le y) \approx \frac{\#\left(j = 1, \dots.., T | r_{p1} \le y\right)}{T}$$

Therefore, we compute the $VaR_\alpha(x)$ and $CVaR_\alpha(x)$ of portfolio returns as follows:

$$VaR_\alpha(x) \approx -r_{\mathrm{p}\lfloor\alpha\mathrm{T}\rfloor}^{\mathrm{sorted}}$$

$$CVaR_\alpha(x) \approx -\frac{1}{\alpha\mathrm{T}} \sum_{\mathrm{j}=1}^{\lfloor\alpha\mathrm{T}\rfloor} r_{\mathrm{p\,j}}^{\mathrm{sorted}}$$

where α is a specified significance level and $\mathrm{r}_{\mathrm{p\,j}}^{\mathrm{sorted}}$are the sorted portfolio returns that satisfy

$$r_{p\,1}^{sorted} \le r_{p\,2}^{sorted} \le \cdots r_{p\,j}^{sorted} \le \cdots \le r_{p\,j}^{sorted}$$

Using time series observation, from (4) the approximation of the partial derivatives $CVaR_\alpha(x)$ for each asst *i* becomes:

$\frac{\partial CVaR_\alpha(x)}{\partial x_i} \approx -\frac{1}{\alpha\mathrm{T}} \sum_{\mathrm{k}=1}^{\lfloor\alpha\mathrm{T}\rfloor} r_{\mathrm{k\,i}}^{\mathrm{sorted}} \;\; \forall$ *i*=1,...,n

and then the total risk contribution of asset i is

$$TRC_i^{CVaR_\alpha(x)}(x) = x_i \frac{\partial CVaR_\alpha(x)}{\partial x_i} \approx -\frac{1}{\lfloor \alpha \mathrm{T} \rfloor} x_i \sum_{\mathrm{k}=1}^{\lfloor \alpha \mathrm{T} \rfloor} r_{\mathrm{k\,i}}^{\mathrm{sorted}}$$

in which $r_{\mathrm{k\,i}}^{\mathrm{sorted}}$ are the pertinent returns of asset i to the sorted portfolio returns. This method was suggested by Stefanovits (2010) in his master thesis, where he applies the equally risk contribution in case of standardized multivariate distribution, using a Gaussian kernel estimation. He implemented Risk Parity approach to Expected Shortfall assuming normally or t-student data in a parametric approach. In this paper we will take in consideration only the time series analysis.

The Matlab code for the computation, we mention with gratitude that we start from the work of (Moussaoui Farid), in which he applies Risk Parity at standard deviation in a form of equally risk contribution and in the same way we apply it to Conditional value at risk by adapting the code. The method is to minimize the distance between the total risk contributions in a large-scale optimization. Since the code is long, we are not putting it as an appendix.

For the diversification measure we measure the following. Consider a portfolio x = (x_1, x_2,.....,x_n) satisfying the budget constraint $\sum_{i=1}^{n} x_i = 1$ with short sales not allowed ($x_i \geq 0$) . The first naive diversification measure is the Herfindal index:

$$D_{Her} = 1 - xx'$$

which takes the value 0 if the portfolio is concentrated in one asset and the maximum value $1 - \frac{1}{n}$ for the equally weighted (or naive) portfolio.

For only strategies $x_i \geq 0$, we introduce the measure proposed by Bera and Park (Bera, Park ,2004).

This diversification measure can be interpreted as the probability of each weight measure in terms of entropy:

$$D_{BP} = -\sum_{i=1}^{n} x_i \log(x_i) = \sum_{i=1}^{n} x_i \log\left(\frac{1}{x_i}\right)$$

The D_{BP} takes value between 0 (fully concentrated in one asset) and log(n) for the naive portfolio.

Another index of diversification based on the weights that compose the portfolio has been proposed by Hannah and Kay:

$$D_{HK}^{\alpha} = -\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\frac{1}{\alpha-1}}$$

For all $\alpha > 0$. It is easy to verify that D_{HK}^{2}=$D_{Her} - 1$.

These three quantities represent diversification only in terms of capital invested and do not take into account that assets contribute differently to the total portfolio volatility.
Another useful index for estimating transaction costs, is the turnover of the portfolio:
$TO = \sum_{i=1}^{n} |x_i^{t+1} - x_i^t|$,
where x_i^t denotes the weight of asset i at time t.

3. Results
In this case we show a mixed portfolio composed with 26 stocks selected from the DAX30, 9 government bonds and 2 commodities (silver and gold) see Fig.1 . We choose a different frequency of data (weekly) and consider the period from January 2000 to December 2013 for the above assets. We do the same analysis for the mixed portfolio creating a rolling time window of 4 years in sample period (208 weekly observations data, in this case, to guarantee the convergence of the model) and rebalancing every month (4 week out of sample). The level of the Conditional Value at Risk is 10% for a better approximation.

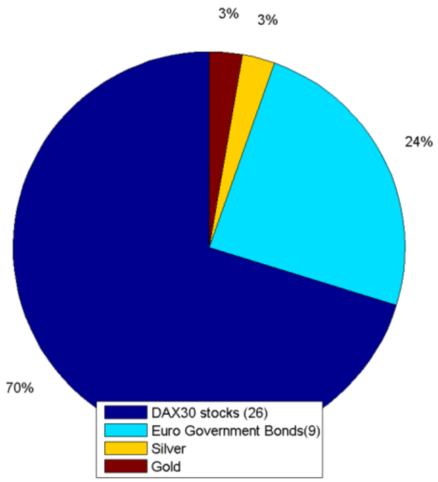


Fig.1 The chart of the mixed portfolio composure

Similar to the method of 60/40 composition introduced at the very beginning by (Qian, 2005) which is the first creation of the Risk Parity strategies, these portfolios not only are well diversified but also outperform the other models. The numerical approximation in this case is applied to weekly observation, so it requires a larger period of the in-sample period, in order for the model to converge. The Risk Parity with Conditional Value at Risk is measured at level of 10% and it perform better that the other models in terms of compound return and Sharpe ration S_{σ}.

Table 1. The performance of the mixed portfolio.

	R.P. -STD	M-V	R.P. CVaR N.	R.P. CVaR	CVaR (10%)	Naive
μ(%)	0.1142	0.0990	0.1059	0.1149	0.0881	0.119
μ_{ann} (%)	6.1136	5.2819	5.6559	6.1508	4.6878	6.397
μ^c(%)	72.7785	65.8115	69.4239	76.1627	56.5337	64.88
σ (%)	9.5905	4.2540	6.7611	7.8331	4.4818	15.36
S_σ	0.6375	1.2416	0.8365	0.7852	1.046	0.4163

When we created a mixed portfolio selecting among stocks from DA30, 26 stocks, 9 government bonds, and 2 commodities (silver & gold). Similar to the method of 60/40 composition introduced at the very beginning by (Qian, 2005) which is the first creation of the Risk Parity strategies, these portfolios not only are well diversified but also outperform the other models. The numerical approximation in this case is applied to weekly observation, so it requires a larger period of the in-sample period, in order for the model to converge. The Risk Parity with Conditional Value at Risk is measured at level of 10% and it perform better that the other models in terms of compound return and Sharpe ration S_σ.

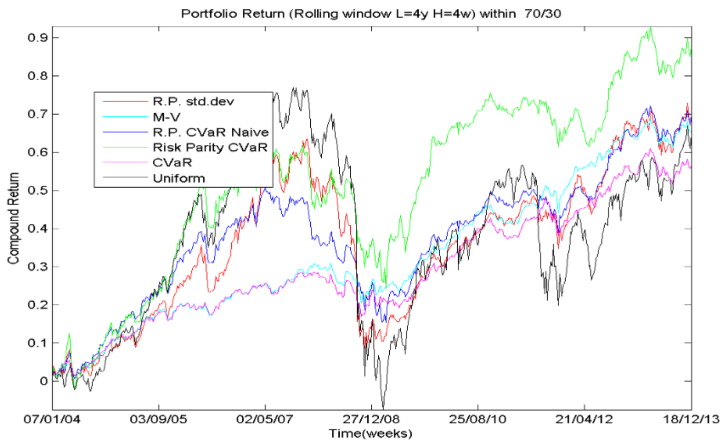
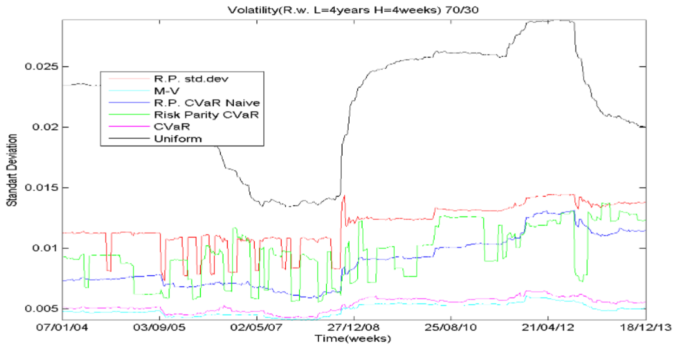


Figure 2. The compound returns of the portfolios in the out of sample period

The Risk Parity with CVaR performs better compared to the other model, especially during the 2018 crisis and start gaining more after it. The Mean Variance and the CVaR are at its minimum risk, although they have an increase trending after the 2008 crisis.

To measure the riskiness of the portfolios, we can use the volatility (standard deviation) or the Conditional value at risk at each end of the holding period.



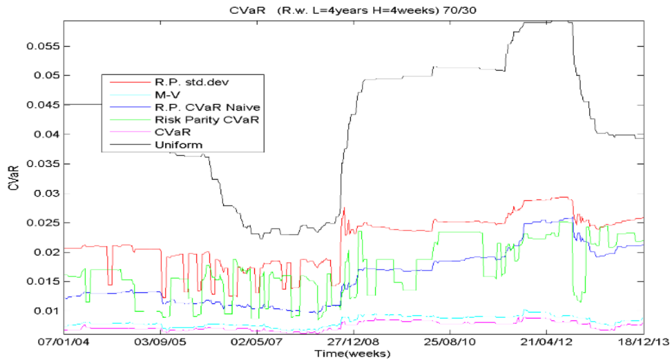


Fig.3 The volatility and the CVaR of the portfolios

The riskiness in terms of volatility and CVaR of the Risk parity models is in between the Naïve and the CVaR and Mean variance models. There is no significant difference between risk parity with CVaR and the standard deviation.

Considering only a commodities market as elements of our portfolio, we will have a problem in high volatility, for that there is a need to study particular models to assign capitals invested in commodities. For this if we use models like the Markowitz model, we will have higher drawdowns if the market moves in decreasing direction, due to the higher concentration. The Risk parity model are more suitable if the set of selection in relatively small (10-15 assets) since they take significative proportion of each weight. For daily frequency data, the Risk Parity model with CVaR converges and the approximation is quite satisfying, in a limited number of iterations. The last to discuss is the diversification and the portfolio turnover.

Table 2 The average portfolio turnover:

RP-Std M-V RP-CVaR Naive RP-CVaR CVaR
Average T-O(%) 0.1985 0.8762 0.3166 0.0387 0.8449

Mean Variance and CVaR have the highest portfolio turnover. Even if we show it graphically (fig. 4) we notice that Risk Parity with CVaR have the lowest turnover, thus less cost of transaction in case of buying or selling the assets.

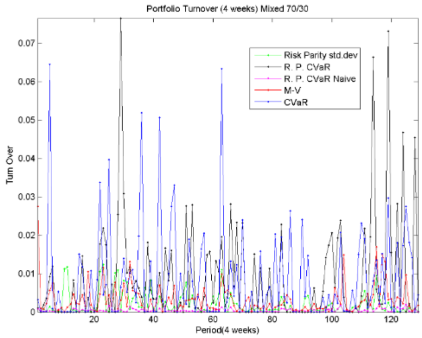


Fig.4 The portfolio turnover in each recalibration

For the Diversification we show the Bera Park Index, the herfindal index, and the number of assets selected. The fist part of the Fig.5 shows the Bera Park index, and it is easy to see that the risk parity models have higher values, thus better diversification and less concentration. In the same logic we have the Herfindal Index, in which Mean Variance and CVaR are focused in a small number of assets. In the bottom graph is shown the number of assets that changes significantly from each model.

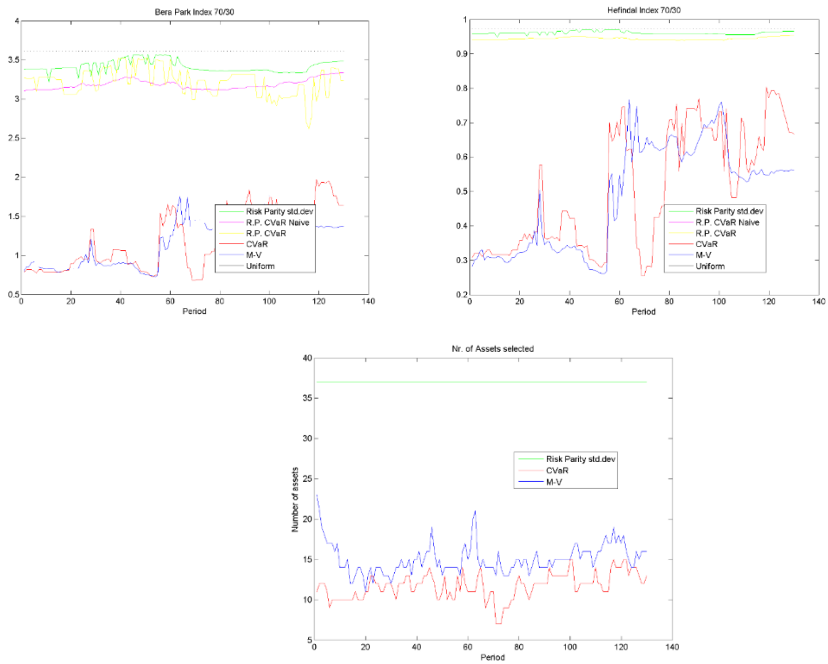


Fig.5 The Bera Park Index/ The herfindal index / No of assets in the selection

4. Conclusion

In this paper we introduced the numerical approximation method for using the Risk Parity with Conditional Value at Risk in the portfolio selection. We gave the needed conditions for the calculation of the partial derivatives in the process for the formulation of the Risk Parity. After that we showed the required condition for the numerical approximation.

The Risk Parity portfolio with Expected shortfall or Conditional Value at Risk will have a linear function as measure of risk, in the optimization, compared to the Risk Parity used mostly in literature (standard deviation). The time of the computation in the same condition (using the same machine) is sensitively reduced.

From the performance point of view, the Risk Parity with Conditional Value at risk in most cases is similar, and not with a significative difference with the Risk Parity with standard deviation. To prove this, we choose a mixed portfolio with stocks, bonds and commodities, the performance of the Risk Parity with Conditional Value at Risk holds better during the impact of 2008 financial crisis and recover faster in the time after it in terms of compound return. This portfolio is diversified in the initial set of possible choices at the beginning, when we apply the risk parity model, we have a benefit in the well diversification.

In conclusion, the risk parity model is a good tradeoff between the minimum risk models and the naïve portfolios, for medium average portfolios. With the numerical computation it is better in case we have a high frequency, like daily observations, to have a better approximation. Using the Conditional Value at Risk, we might pass to a coherent measure, and use its benefits.

References

Acerbi C., Tasche D. (2002). On the coherence of expected shortfall. *Journal of Banking & Finance 26 1487*.150,

Artzner P., Delbaen F. (1999). Coherent measures of risk. *Journal of Mathematical Finance 9*:203-228.

Andersson F., Mausser H., Uryasev S. (2000) Credit Risk Optimization with Conditional Value at Risk Criterion. *Journal of Mathematical Programming*.

Bacon C.R. (2008), Practical Portfolio Performance Measurement and Attribution, *vol. 546,* Wiley,

Bertsimas D., Lauprete G.,Samarov A.,(2004). Shortfall as a risk measure: properties and optimization. *Journal of Economic Dynamics and Control, 28, 7*, 1353-1381.

Bera A.K., Park S.Y. (2004). Optimal portfolio diversification using maximum entropy. Working Paper.

Black F., Litterman R., (1990). Asset Allocation: Combining Investors Views with Market Equilibrium *Fixed Income Research*, Goldman, Sachs & Company, Biglova A., Ortobelli S., Rachev S.T., Stoyanov S. (2004). Different Approaches to Risk Estimation in Portfolio Theory. *Journal of Portfolio Management 31*. 103-112.

Caporin M., Lisi F., Janin M., (June 2012). A survey on Four Families of Performance Measures. Working papers series.

Colucci S., (2013). A quick introduction to quantitative models that discard estimation of expected returns for portfolio construction. Working Paper.

Dowd K., (2000). Adjusting for Risk: An Improved Sharpe Ratio., I*nternational Review of Economics and Finance 9*: p.209-222.

Fama E.(1965), The behavior of Stock Market Prices, *Journal of Business 38*, 34-105.

Konno H., Yamazaki H. (1991). Mean-Absolute Deviation Portfolio Optimization Model and Its Applications to Tokyo Stock Market., *Management Science 37*, 519-531

McNeil A.J., Frey R. (2000). Estimation of Tail-Related Risk Measures for Heteroscedastic Financial Time Series: An Extreme Value Approach. *Journal of Empirical Finance 7*,271-300,

Maillard S., Roncalli T., Teiletche J. (2009), On the properties of equally weighted risk contributions portfolios. *Journal of Portfolio Management*

Maillard S., Roncalli T., Teiletche J. (2012), Managing Risk Exposures using Risk Budgeting Approach. Working Paper.

Markowitz H.M. (1952), Portfolio selection. *Journal of Finance, 7*:77. 91,.

Markowitz H., (1987). Mean-Variance analysis in portfolio choice and capital markets, *Basil Blackwell*

Merton R.C. (1980) On estimating the expected return on the market", Journal of Financial Economics, 8(4):323. 361.

Qian E., (2005). Risk Parity Portfolios. *Pan Agora Asset Management*.

Risk Metrics (1996). Technical Document. *Morgan Guaranty Trust Company of New York*,.

Rockfellar R.T., Uryasev S. (2000). "Optimization of Conditional Value ar Risk", The Journal of Risk.

Sharpe W. (1966). Mutual Fund Performance., *Journal of Business 39*: 119-138.

Stefanovits D. (2010).Equal Contributions to Risk and Portfolio Construction. Master Thesis, ETH Zurich, 33,

Tasche D. (2000), Conditional expectation as quantile derivative. Risk Contributions and Performance Measurement. Preprint, *Department of Mathematics*, TU-Munchen,