Research Article

Some empirical results using block bootstrap in estimating the coefficients of a periodic autoregressive model

Lorena Margo Zeqo
Fan S. Noli University, Korca, Albania

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Abstract

The bootstrap proposed by Efron (1979) resulted a useful method in estimating the distribution of an estimator or a test statistic by resampling the data in the case of independent and identically distributed observations. Although it was not as effective in the case of dependent data as in the case of independent and identically distributed data, an adaptation was obtained using the block bootstrap. The block bootstrap consists in dividing the data into blocks of observations and then resampling these blocks with replacement. When resampling periodic data, we must take in consideration the periodicity present.
Periodically correlated time series and in particular those related with PAR processes have been object of many recent studies due to numerous applications in real data problems. The aim of this paper is to use a block bootstrap procedure proposed, Block Bootstrap of the Residuals, in the case of PAR (Periodic Autoregressive) models. The results obtained in the case of estimating the coefficients in a PAR model studied are very good and are characterized by small values of Bias, Mean Squared Error and Standard Deviation. Also the bootstrap estimations obtained are closer to the true values than the usual classic point estimations.

Keywords: Block bootstrap, periodicity, Periodic Autoregressive (PAR) models, estimation, confidence interval, resampling.

1. Introduction

Periodically correlated (or cyclostationary) processes are random processes that have a periodic structure but are also random. Many processes encountered in nature and in human activity possess periodic properties such as in meteorology, communication systems, econometrics etc.
Periodically correlated random processes are random processes in which a periodic rhythm exists in the structure that is generally more complicated than periodicity in the mean function. An important class of periodically correlated sequences are the

periodic autoregressive (PAR) models (Hurd & Miamee, 2007).
Periodically correlated time series and in particular those related with PAR processes have been object of many recent studies due to numerous applications in real data problems.
Bootstrap introduced first by Efron (1979) resulted to be an important method for estimating the distribution of an estimator or test statistic by applying the resampling of the data. An adaptation of bootstrap in the case of stationary time series is the block bootstrap. Block bootstrap consists in dividing the data into blocks of observations and then resampling these blocks with replacement (e.g., Moving Block Bootstrap (MBB) proposed by Künsch (1989) and Liu and Singh (1992).
For the time series with a seasonal component, Politis (2001) proposed the Seasonal Block Bootstrap (SBB) that is a version of Kunsch's (1989) block bootstrap with blocks whose size and starting points are integer multiples of the period and also Dudek et. Al (2014) proposed a modification of the block bootstrap, the Generalized Seasonal Block Bootstrap (GSBB) that is suitable for periodic time series with fixed length periodicities of arbitrary size as related to block size and also to the sample size.
A block bootstrap procedure is applied, Block Bootstrap based on residuals (BBR) in estimating the coefficients in a PAR (2) time series obtained from a Periodic Autoregressive of second order model based on Fourier representation of the Periodic coefficients of the model (Dudek, Hurd & Wojtowicz, 2016). The method proposed, Block bootstrap based on residuals is characterized from a good performance in periodically correlated time series models and also in real data time series (Margo Zeqo, 2023).

2. Objectives

The use of bootstrap methods in parameter estimation has been the focus of many studies in recent years. But not many studies have been done in the case of periodic autoregressive models in using bootstrap methods in estimating the parameters of the model. The main objective of this paper is to explore the performance of a block bootstrap procedure proposed, the Block bootstrap based on residuals in the point and interval estimation of the coefficients obtained from the Fourier decomposition in a Periodic Autoregressive model of the second order.

3. Methodology

- **Data in this study**

In this study a PAR(2) model is considered, with non-zero coefficients a_{11}= 0.6, a_{12}= 0.2 and a_{21}= -0.5 with a constant shock. A simulation is conducted based on a sample of size 120 of a PAR (2) model with period 12 and the interest is in applying the proposed block bootstrap method in estimating the coefficients and comparing the results with classic point estimations and with real values.

- **Bootstrap methodology**

Efron's bootstrap (1979) resulted to be an important method for estimating the distribution of an estimator or test statistic by applying the resampling of the data. In the case of the independent and identically distributed observations (i.i.d) this method gives good results, but in the case of dependent observations, like time series, the i.i.d bootstrap gives incorrect answers (Singh, 1981).

The block bootstrap resulted to be an important and well-known method for implementing the bootstrap with the time series data. This method consists in dividing the data into blocks of observations and then resampling these blocks with replacement. Moving Block Bootstrap (MBB) proposed by Künsch (1989) and Liu and Singh (1992), is considered a good method for bootstrapping the data designed especially for stationary time series.

For the time series with a seasonal component, Politis (2001) proposed the Seasonal Block Bootstrap (SBB) that is a version of Kunsch's (1989) block bootstrap with blocks whose size and starting points are integer multiples of the period. Also Dudek et. Al (2014) proposed a modification of the block bootstrap, the Generalized Seasonal Block Bootstrap (GSBB) that is suitable for periodic time series with fixed length periodicities of arbitrary size as related to block size and also to the sample size and because the block size can be chosen independently from the period, the usual asymptotic considerations for block size choice avoid the inconsistency problems of other methods and the problems that associate the Seasonal Block Bootstrap of Politis.

The procedure proposed, Block bootstrap based on residuals (BBR) is based on the idea from the Residual-based block bootstrap of Paparoditis and Politis (2003) and is also adapted in the case of periodic models (Margo & Ekonomi, 2017). BBR consists in using block bootstrap resampling in the series of the residuals considering the periodicity of the series and then obtaining the bootstrap sample by considering the estimated coefficients and the resampled time series of the residuals.

- **PAR (Periodic Autoregressive Models) and PARMA (Periodic Autoregressive Moving Average) models**

In several studies there are encountered data collected in time that exhibit cyclical variations. One class of such models consists of time series with periodically varying dependence structures. The periodicity could be in the mean, the variance, but also in the model parameters such as with periodic autoregressive (PAR) models that play a central role in this class of models (Hurd & Pipiras, 2020). A sequence of random variables of the second order, X_t, with zero mean, is called periodic autoregressive of order p, PAR(p) if it satisfies the following condition:

$$X_t = \sum_{j=1}^{p} \varphi(t) X_{t-1} + \sigma(t)\varepsilon_t$$

where is ε_t is an orthonormal sequence and $\varphi(\mathrm{t}) = \varphi(\mathrm{t+T}), \sigma(\mathrm{t}) = \sigma(\mathrm{t+T})$.

A sequence of random variables of the second order, X_t, with zero mean, is called periodic autoregressive of the first order, PAR(1), if:

$$X_t = \varphi(\mathrm{t})\,\mathrm{X}_{t-1} + \sigma(\mathrm{t})\varepsilon_t$$

where $\varphi(\mathrm{t}) = \varphi(\mathrm{t+T}), \sigma(\mathrm{t}) = \sigma(\mathrm{t+T})$ are real and ε_t is an orthonormal sequence. The sequence $\{\sigma(\mathrm{t})\varepsilon_t\}$ is the shock sequence. For $\sigma(\mathrm{t}) \equiv \sigma$, the model is considered as a PAR(1) with constant variance shock (Hurd & Miamee, 2007). In the case of PAR (p), the number of parameters is of the order of pT, a number which is again large compared to the sample size. A second order stochastic sequence X_t is called PARMA(*p*,*q*) with period if it satisfied the following condition:

$$X_t = \sum_{j=1}^{p} \varphi_j(\mathrm{t})\,\mathrm{X}_{t-j} + \sum_{k=1}^{q} \theta_k(\mathrm{t})\varepsilon_{t-k} + \sigma(\mathrm{t})\varepsilon_t$$

where $\varphi_j(\mathrm{t}) = \varphi_j(\mathrm{t+T}), \theta_k(\mathrm{t}) = \theta_k(\mathrm{t+T}), \sigma(\mathrm{t}) = \sigma(\mathrm{t+T})$ për çdo $j = 1, 2, ..., p$, $k = 1, 2, ..., q$ are periodic coefficients and ε_t is an orthonormal sequence. In many cases we denote $\theta_0(\mathrm{t}) = \sigma(\mathrm{t})$ PARMA(*p*,*q*) processes in total have (*p*+*q*+1)*T* parameters, that is, a relatively large number of parameters to be evaluated. An alternative parametrization for PARMA models was suggested by Jones dhe Brelsford (1967) based in Fourier series in order to reduce the number of necessary parameters (previously was suggested for PAR models):

$$\varphi_j(\mathrm{t}) = \mathrm{a}_{j,1} + \sum_{m=1}^{[\mathrm{T}/2]} a_{j,2m} \cos(2\pi\,\mathrm{mt/T}) + \sum_{m=1}^{[\mathrm{T}/2-1]} a_{j,2m+1} \sin(2\pi\,\mathrm{mt/T}) \quad \mathrm{j} = 1, 2, ..., \mathrm{p}$$

$$\theta_k(\mathrm{t}) = b_{\mathrm{k},1} + \sum_{m=1}^{[\mathrm{T}/2]} b_{\mathrm{k},2m} \cos(2\pi\,\mathrm{mt/T}) + \sum_{m=1}^{[\mathrm{T}/2-1]} b_{\mathrm{k},2m+1} \sin(2\pi\,\mathrm{mt/T}) \quad k = 0, 1, 2, ..., \mathrm{q}$$

where $\theta_0(\mathrm{t}) = \sigma(\mathrm{t})$. Reducing the number of parameters can be achieved by limiting the number of frequencies in the Fourier series. This alternative parametrization of PARMA models can in many cases reduce the number of parameters needed to represent such a model. The study is focus mainly on the Maximum Likelihood method, applied not to the parameters themselves: $\{\varphi_j(\mathrm{t}), \mathrm{j}=1,2,\ldots,\mathrm{p};\ \theta_k(\mathrm{t}), \mathrm{k}=1,2,\ldots,\mathrm{q};\ \sigma(\mathrm{t})\}$, but to the coefficients in their Fourier

decompositions (Dudek, Hurd & Wojtowicz, 2016).

- **R programming language**

In this study R programming language is used. R is a free software environment for statistical computing and graphics. This programming language is used to implement the algorithms proposed and to obtain and present the results. *Gdata* package is used and in the case of Periodic Autoregressive time series models (PAR) and also Periodic Autoregressive Moving Average models (PARMA) there are several suitable packages in R. *Partsm* package introduced by L´opez-de-Lacalle (2005) can be used to check for periodicity in the data, fit a periodic autoregressive model of order p, PAR(p), select the periodic autoregressive lag order parameter, test for periodic integration, fit a periodically integrated autoregressive model up to order 2, PIAR, as well as to perform out-of-sample forecasts. *PerARMA* package in R helps in the case of identification, model fitting and estimation for time series with periodic structure and also has additionally procedures for simulation of periodic processes and real data sets are included.

4. **Results and discussion**

With intention to explore the performance of the block bootstrap in the case of the time series data from a Periodic Autoregressive model we considered a N=120 sample simulation of a PAR (2) model with period 12 with non-zero coefficients a_{11}= 0.6, a_{12}= 0.2 and a_{21}= -0.5 with a constant shock. R programming language is used to construct the codes and also the before mentioned R packages. In the figure below the plot of the series generated from the model is presented with N=120.

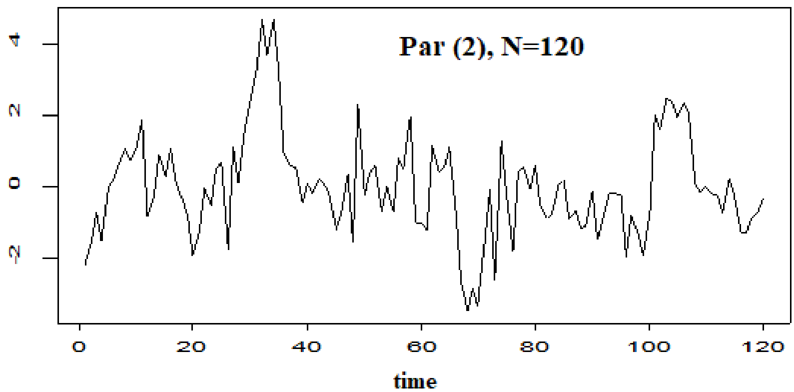

Fig. 1 Time series generated from the PAR(2) model, *N*=120 (sample size) and with coefficients a_{11}=0.6, a_{12}=0.2 , a_{21}= -0.5 and the period *T*=12.

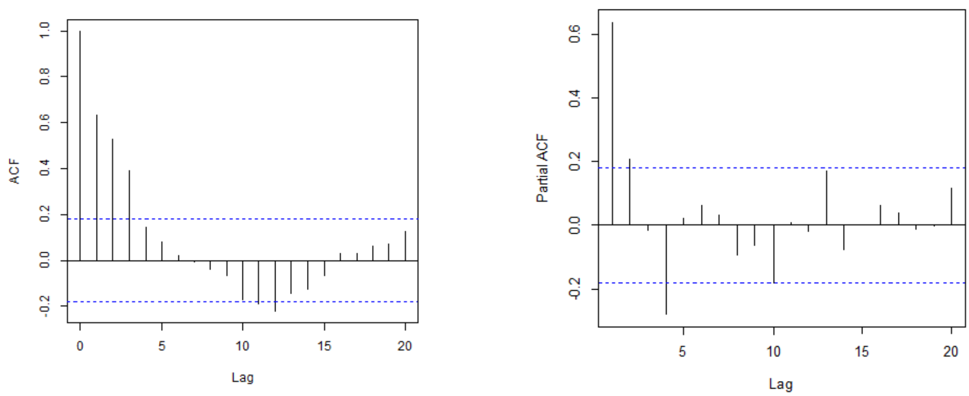

Fig. 2 ACF plots and PACF plots for the PAR(2) time series studied.

The proposed bootstrap method is applied on the series of residuals, after estimating the parameters. The simulations are performed using the programming language R. The period chosen is T=12, as the block length b=5 is chosen and the simulations using 300 bootstrap replications and 400 Monte Carlo trials are performed. The bootstrap estimations were found not only for the coefficients a_{11}, a_{12} and a_{21}, and b_{01}, since we have the model with constant shock and the performance of the bootstrap method is seen also in the case of this coefficient. For the resulting bootstrap estimations the Bias, Standard Deviation and Mean Squared Error (MSE) are calculated.

Table 1 Bootstrap estimation, Bias, Mean Squared Error (MSE) and Standard Deviation of bootstrap estimation of parameters.

	Bootstrap estimation	**Bias**	**MSE**	**Standard Deviation**
$\mathbf{a_{11}}$	0.5700216	-0.0299784	0.01732002	0.1283216
$\mathbf{a_{12}}$	0.1960258	-0.0039742	0.01913036	0.1384452
$\mathbf{a_{21}}$	-0.5165039	-0.0165039	0.02458061	0.1561423
$\mathbf{b_{01}}$	0.9783396	-0.02166038	0.004686919	0.06503333

In Table 2, we present the true values of the parameters as well as the point estimate while in Table 3 we present the 95% percentile bootstrap confidence intervals obtained for the parameters.

Table 2 True values of the parameter and point estimation

	True values	**Point estimation**
$\mathbf{a_{11}}$	0.6	0.6290484
$\mathbf{a_{12}}$	0.2	0.2905193

a_{21}	-0.5	-0.5498256
b_{01}	1	0.788633

Table 3 95 % percentile bootstrap confidence intervals for the parameters

	The percentile bootstrap confidence intervals	
	Low	Upper
a_{11}	0.3286633	0.814919
a_{12}	-0.5137198	0.4273037
a_{21}	-0.7908708	-0.2245649
b_{01}	0.8486474	1.10763

Fig. 3 is used to realize a comparison between point estimations, bootstrap estimations and true values of the parameters:

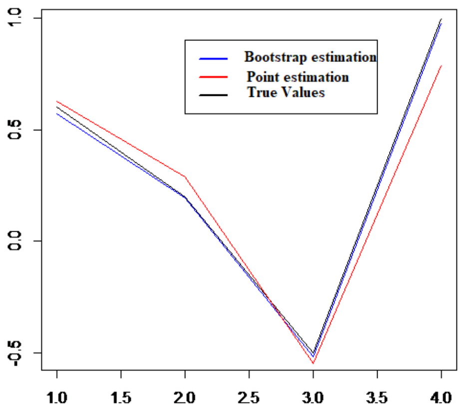

Fig. 3 Graphical representation of the bootstrap estimations, point estimations, and true values for the parameters in the Fourier decomposition of the coefficients of the considered PAR model.

5. Conclusion

Based on the results, we notice a fairly good performance of the bootstrap method used, compared to the usual point estimation.

The bootstrap estimations obtained are closer to the true values than the classical point estimations and also the bootstrap estimations are characterized by small values of Bias, Mean Squared Error and Standard Deviation.

The block bootstrap method proposed resulted a very good method in estimating the coefficients of the PAR model and it is considered an appropriate method in future studies in the case of estimating the parameters in periodic time series models.

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